A MODIFICATION OF THE ITERATION METHOD OF M. E. SHVETS

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UDC 518.61

A modification of the well-known method of M. E. Shvets [1, 2] is proposed, whereupon the method becomes iterational – interpolational.

Iteration [1-3] and interpolation [4-6] methods of solving problems of mathematical physics are known at this time.

The common disadvantage of iteration methods, including that of M. E. Shvets, is that they can be used only in solving those problems where the desired function is itself given on the boundaries of the system being studied. Otherwise, the convergence of the approximations become weak, and in a number of cases the sequence of approximations is generally not constructed successfully by means of known iteration schemes.

Interpolation methods do not have the above-mentioned disadvantage, but, when they are used, it is necessary to establish from a priori considerations the profile of the required function in one of the independent variables; this affects the accuracy of the method radically. This disadvantage is cancelled by increasing the number of "free" parameters, but in consequence the volume of the computational work grows considerably.

An iteration – interpolation method of solving problems of mathematical physics is proposed below.

1. Let us present a logical scheme of the method and let us prove the convergence of successive approximations for the solution of the Cauchy problem

$$\frac{\partial}{\partial x}\left(x^{\mu}\frac{\partial\theta}{\partial x}\right)=x^{\mu}\frac{\partial\theta}{\partial\tau},\qquad \alpha_{\mu}\int_{0}^{\infty}\theta(x)\,x^{\mu}dx=Q<\infty, \tag{1}$$

$$\theta|_{\tau=0}=\begin{cases}\infty & \text{for } x=0,\\ 0 & \text{for } x\neq 0,\end{cases} \tag{2}$$

whose solution determines the fundamental solution of the heat conduction equation [7].

Let us consider an intense change in the temperature θ to occur in a temperature boundary layer of finite thickness $\Delta = \Delta(\tau)$, and let us introduce the quantity $\theta_0 = \theta_0(\tau)$, the temperature at x = 0, unknown in advance. Then taking account of the symmetry of the heat propagation process, condition (2) can be replaced by the conditions

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0}=0,\quad \theta|_{x=0}=\theta_0(\tau),\quad \theta|_{x=\Delta}=0,\quad \left.\frac{\partial\theta}{\partial x}\right|_{x=\Delta}=0, \tag{3}$$

$$\Delta(0)=0,\quad \theta_0(0)=\infty,\quad Q=\alpha_{\mu}\int_{0}^{\Delta}x^{\mu}\theta(x)\,dx. \tag{4}$$

The last condition (3) is a result of the assumption that the temperature varies in a layer of finite thickness, and that the temperature gradient is continuous.

V. V. Kuibyshev Tomsk State University. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 19, No. 1, pp. 84-93, July, 1970. Original article submitted March 31, 1969.

TABLE 1. Comparison of the Exact and Approximate Solutions of the Boundary-Value Problem (34), (35)

τ	x				
	0,1	0,2	0,3	0,4	0,5
0,25	0,041919	0,073569	0,096149	0,109595	0,114060
	0,041688	0,073699	0,096328	0,109805	0,114280
0,5	0,044713	0,079455	0,104249	0,119117	0,124072
	0,044725	0,079476	0,104279	0,119153	0,124109
1	0,044998	0,079996	0,104995	0,119994	0,124993
	0,044998	0,079996	0,104995	0,119994	0,124994

We take the function

$$\theta^0 = \theta_0 (1 - y^2), \quad y = \frac{x}{\Delta}, \tag{5}$$

which satisfies the first three conditions (3) as the zero approximation. Substituting (3) in the right side of (1) and integrating the result of substituting x twice taking account of the first two conditions in (3), we obtain an expression for the first approximation

$$\theta^{(1)} = \theta_0 + \frac{\Delta^2 \dot{\theta}_0}{2(\mu + 1)} y^2 + \frac{\Delta^2}{4(\mu + 3)} \left(\frac{2\theta_0 \dot{\Delta}}{\Delta} - \dot{\theta}_0 \right) y^4. \tag{6}$$

From the demand that (6) satisfy the last two conditions in (3), we obtain two ordinary differential equations to determine the quantities Δ and θ_0

$$\dot{\theta}_0 \Delta^2 (\mu + 5) + 2\Delta\dot{\Delta}\theta_0 (\mu + 1) + 4(\mu + 3)(\mu + 1)\theta_0 = 0,$$
$$\frac{d}{d\tau} (\theta_0 \Delta^{\mu+1}) = 0. \tag{7}$$

If we decided on just the first approximation, then by solving (7) taking (4) into account, we will find

$$\Delta_1 = \sqrt{8\tau}, \quad \theta_0 = \frac{Q(\mu + 1)(\mu + 3)(\mu + 5)}{8\alpha_\mu (2\sqrt{2\tau})^{\mu+1}}. \tag{8}$$

To determine the next approximation, $\dot{\theta}_0$ and $\dot{\Delta}$ should be eliminated from (6) by using (7), and we consequently find

$$\theta^{(1)} = \theta_0 (1 - y^2)^2, \quad y = \frac{x}{\Delta}, \tag{9}$$

after which we find θ_0 and Δ analogously in a second approximation.

Repeating the process in the order indicated, we obtain

$$\theta^{(n)} = \theta_0 (1 - y^2)^{n+1}, \quad \Delta_n = 2\sqrt{(n + 1)\tau}. \tag{10}$$

It is easy to see that it is impossible to construct a sequence of approximations in this case by using the classical method of M. E. Shvets.

Using the first of conditions (4), we obtain an expression for θ_0 in the n-th approximation

$$\theta_0^{(n)} = \frac{2Q}{\Delta_n^{\mu+1} \alpha_\mu \int_0^1 (1 - y^2)^{\mu+1} dy} = \frac{2Q\Gamma\left(n + \frac{\mu + 5}{2}\right)}{\alpha_\mu (2\sqrt{(n + 1)\tau})^{\mu+1} \Gamma\left(\frac{\mu + 1}{2}\right) \Gamma(n + 2)}. \tag{11}$$

TABLE 2. Comparison of the Exact and Approximate Solutions of the Boundary-Value Problem (39), (40)

τ_1	x				
	0,1	0,2	0,3	0,4	0,5
0,25	0,049999 0,050026	0,100810 0,099393	0,147502 0,145737	0,177485 0,177676	0,187500 0,187691
0,5	0,000000 —0,000165	0,000000 0,000097	0,000000 0,000718	0,000000 —0,002137	0,000000 —0,005451
1	—0,090001 —0,086543	—0,160744 —0,160508	—0,209999 —0,211478	—0,239767 —0,239784	—0,250000 —0,249451

It is easy to see that

$$\lim_{n\to\infty}\theta_0^{(n)} = \frac{Q}{(2\sqrt{\pi\tau})^{\mu+1}},$$

$$\lim_{n\to\infty}\theta^{(n)} = \lim_{n\to\infty}\left[1-\frac{x^2}{4(n+1)\tau}\right]^{n+1} = \frac{Q\exp-\dfrac{x^2}{4\tau}}{(2\sqrt{\pi\tau})^{\mu+1}}. \tag{12}$$

Therefore, the sequence of approximations reduces to the known fundamental solution of the heat conduction equation [7].

2. Let us note that if the classical method of M.E. Shvets [1, 2] is iterational in character, the proposed method is one of iteration–interpolation. The accuracy of the solution obtained by using this method can be increased both by increasing the number of iterations, and by introducing a greater number of "free" parameters to be determined.

Let us refine the solution of the Cauchy problem (1), (2) for $\mu = 0$. To this end, let us partition the interval $0 < x < \Delta$ into k + 1 parts. Let us give the zero approximation in the i-th interval as

$$\theta_i^0 = A_i x + B_i,\quad A_i = \frac{\theta_i - \theta_{i-1}}{x_i - x_{i-1}},\quad B_i = \frac{\theta_{i-1}x_i - \theta_i x_{i-1}}{x_i - x_{i-1}}, \tag{13}$$

where $\theta_i = \theta(x_i, \tau)$, $x_i = m_i\Delta(\tau)$, the quantity m_i is a constant, where $0 < m_i < 1$ and $m_0 = 0$, $m_{k+1} = 1$.

Substituting (13) into the right side of (1) for $\mu = 0$ and integrating twice with respect to x while taking account of the boundary conditions

$$\theta|_{x=x_{i-1}} = \theta_{i-1},\quad \theta|_{x=x_i} = \theta_i, \tag{14}$$

we obtain the first approximation in the i-th interval as

$$\theta_i^{(1)} = \theta_{i-1} + \frac{\dot{A}_i(x^3 - x_{i-1}^3)}{6} + \frac{\dot{B}_i(x^2 - x_{i-1}^2)}{2} + \frac{x - x_{i-1}}{x_i - x_{i-1}}\left[\theta_i - \theta_{i-1} - \frac{\dot{A}_i(x_i^3 - x_{i-1}^3)}{6} - \frac{\dot{B}_i(x_i^2 - x_{i-1}^2)}{2}\right]. \tag{15}$$

To determine $\theta_i = \theta_i(\tau)$ we use the conditions

$$\left.\frac{\partial\theta_1^{(1)}}{\partial x}\right|_{x=0} = 0,\quad \left.\frac{\partial\theta_i^{(1)}}{\partial x}\right|_{x=x_i} = \left.\frac{\partial\theta_{i+1}^{(1)}}{\partial x}\right|_{x=x_i},\quad \left.\frac{\partial\theta_{k+1}^{(1)}}{\partial x}\right|_{x=\Delta} = 0, \tag{16}$$

whose number agrees with the number of unknown "free" parameters. Let us put k = 2, then the unknowns Δ, θ_0, θ_1, θ_2 and $\theta_3 = 0$. We obtain the following equations to determine the unknowns from the conditions (16):

$$\Delta\left[\dot{\theta}_0 m_1 + \dot{\theta}_1 m_2 + (1-m_1)\dot{\theta}_2\right] + \dot{\Delta}\left[m_1(\theta_1 - \theta_2) + m_2\theta_1 + \theta_2\right] = 0, \tag{17}$$

$$2m_1^2\Delta^2\dot{\theta}_0 + m_1^2\Delta^2\dot{\theta}_1 - \Delta\dot{\Delta}m_1^2(\theta_1 - \theta_0) = 6(\theta_1 - \theta_0), \tag{18}$$

TABLE 3. Comparison of the Exact and Approximate Solutions of the Boundary-Value Problem (44), (45)

x	y				
	0,1	0,2	0,3	0,4	0,5
0,1	0,013070 0,013228	0,021210 0,021019	0,025628 0,025745	0,028117 0,028327	0,029043 0,029150
0,5	0,029041 0,029148	0,050070 0,049892	0,063380 0,063631	0,071037 0,071436	0,073672 0,073964

$$3m_1^2\Delta^2\theta_0+\dot{\theta}_1\Delta^2(m_1^2+m_1m_2+m_2^2)+\dot{\theta}_2\Delta^2(3m_2-m_1-m_1m_2-m_2^2)$$
$$-\Delta\dot{\Delta}\left[3(\theta_1-\theta_0)m_1^2+2(\theta_2-\theta_1)(m_1^2+m_1m_2+m_2^2)-3\theta_2m_2(1+m_2)\right]$$
$$=6(\theta_1-\theta_2), \tag{19}$$

$$\dot{\theta}_2\Delta^2(1-m_2)^2-\Delta\dot{\Delta}\theta_2(m_2^2+m_2-2)=6\theta_2. \tag{20}$$

Let us seek the solution of the system (17)-(20) as

$$\theta_0=\frac{p}{\sqrt{\tau}},\quad \theta_1=\frac{q}{\sqrt{\tau}},\quad \theta_2=\frac{r}{\sqrt{\tau}},\quad \Delta=s\sqrt{\tau}, \tag{21}$$

where p, q, r, s are unknown constants.

Upon substituting (21) into (17)-(20), equation (17) becomes an identity, and the remaining three equations transform into algebraic equations which in conjunction with the condition of heat conservation

$$Q=2\left[\int_0^{x_1}\theta_1^{(1)}(x)\,dx+\int_{x_1}^{x_2}\theta_2^{(1)}(x)\,dx+\int_{x_2}^{\Delta}\theta_3^{(1)}(x)\,dx\right] \tag{22}$$

determine the constants

$$q=\frac{Q}{2s}\left\{m_1(g-1)+m_2+l(1-m_2)\right.$$
$$-\frac{s^2}{24}\left[gm_1^2+l(m_1^3-m_1m_2-m_1m_2^2+4m_2^3-5m_2^2+m_2+1)\right.$$
$$\left.\left.+4m_1^3-5m_1^2m_2+m_1m_2^2+m_2^3\right]\right\}^{-1}, \tag{23}$$

$$s^2=\frac{12}{1-m_2-2m_2^2},\quad g=\frac{\theta_0}{\theta_1}=\frac{1+m_2-2m_2^2+2m_1^2}{1+m_2-2m_2^2-m_1^2}, \tag{24}$$

$$l=\frac{\theta_2}{\theta_1}=\frac{1+m_2-m_2m_1-3m_2^2+2m_1^2}{1+m_2-m_1m_2-m_1^2}. \tag{25}$$

If k = 1, then for $m_1 = m_2 = 0$ we obtain from (23)-(25)

$$\theta_0=\theta_1=\theta_2=\frac{Q}{s\sqrt{\tau}},\quad s=\sqrt{12},\quad \Delta=\sqrt{12\tau}. \tag{26}$$

It follows from (12) and (26) that the relative error in the quantity θ_0 equals $\varepsilon = 2.2\%$.

If k = 2 and $m_1 = 0.057894$, but $m_2 = 2/3$, then

$$g=1.012985,\quad l=0.185535,$$

$$\Delta=3.927922\sqrt{\tau},\quad \theta_0=\frac{0.282147Q}{\sqrt{\tau}} \tag{27}$$

and the error in θ_0 does not exceed 0.01%, i.e., the error in ε diminished substantially as the number of "free" parameters increased. This is explained by the fact that the iteration will be more accurate the smaller the domain of its definition.

3. The proposed method is also applicable to the solution of nonlinear problems of boundary-layer theory. As an illustration, let us consider the ignition of a semiinfinite reacting space heated by a plane surface. Mathematically this problem reduces to solving the boundary-value problem

$$\frac{\partial^2\theta}{\partial\xi^2} = \frac{\partial\theta}{\partial\tau} - \exp\theta, \quad \theta|_{\xi=0} = 0, \quad \theta|_{\xi=\Delta} = -\theta_H, \quad \Delta(0) = 0. \tag{28}$$

Here all the notation has been taken from [8] and the boundary-layer thickness $\Delta = \Delta(\tau)$ has been introduced. Let us partition the interval $0 < \xi < \Delta$ into $k + 1$ parts and let us select the linear profile (13) as the zero approximation in the i-th interval.

Taking account of (28) we obtain the first approximation in the i-th interval as

$$\begin{aligned}\theta_i^{(1)} &= \theta_{i-1} + \frac{\dot{A}_i(x^3 - x_{i-1}^3)}{6} + \frac{\dot{B}_i(x^2 - x_{i-1}^2)}{2} \\ &+ \frac{1}{A_i^2}\left[\exp\theta_{i-1} - \exp(A_i x + B_i)\right] \\ &+ \frac{x - x_{i-1}}{x_i - x_{i-1}}\left[\theta_i - \theta_{i-1} - \frac{\dot{A}_i(x_i^3 - x_{i-1}^3)}{6}\right. \\ &\left. - \frac{\dot{B}_i(x_i^2 - x_{i-1}^2)}{2} + \frac{\exp\theta_i - \exp\theta_{i-1}}{A_i^2}\right].\end{aligned} \tag{29}$$

For simplicity, let us take one internal point $x_1 = m\Delta$. Then we obtain a system of two ordinary differential equations to determine Δ and θ_1

$$\begin{aligned}\frac{\dot{\theta}_1\Delta}{3} &+ \frac{\dot{\Delta}}{6}\left[\theta_1 + (2m+1)\theta_H\right] = \left(\frac{1}{A_1} - \frac{1}{m\Delta A_1^2}\right. \\ &\left. - \frac{1}{\Delta A_2^2(1-m)} - \frac{1}{A_2}\right)\exp\theta_1 - \frac{\theta_1 + m\theta_H}{m(1-m)\Delta} \\ &+ \frac{\exp(-\theta_H)}{\Delta(1-m)A_2^2} + \frac{1}{m\Delta A_1^2},\end{aligned} \tag{30}$$

$$\begin{aligned}\frac{\Delta\dot{\theta}_1(1-m)}{6} &+ \frac{\dot{\Delta}(\theta_H + \theta_1)(m+2)}{6} = \frac{\theta_H + \theta_1}{\Delta(1-m)} \\ &+ \frac{\exp\theta_1}{A_2^2\Delta(1-m)} + \left(1 - \frac{1}{A_2\Delta(1-m)}\right)\frac{\exp(-\theta_H)}{A_2}.\end{aligned} \tag{31}$$

If there is no heat evolution from the chemical reaction, then $\exp(-\theta_H)$ and $\exp\theta_1$ should be considered zero, and the system (30), (31) has the solution

$$\theta_1 = -\frac{3m\theta_H}{2}, \quad \Delta = \sqrt{\frac{12\tau}{2 - m - m^2}}. \tag{32}$$

Taking account of (32), we have for the corresponding linear problem

$$\left.\frac{\partial\theta}{\partial\xi}\right|_{\xi=0} = -\frac{3(2-m)\theta_H}{2\sqrt{12(m+2)(1-m)\tau}}. \tag{33}$$

In conformity with [1], for $m = 0$ we obtain the numerical coefficient 0.61 in (33), and 0.576 for $m = 2/5$, which agrees better with the exact value [7], equal to $1/\sqrt{\pi} = 0.564$.

For different θ_H the system (30), (31) with $m = 0.057$ was solved numerically, taking account of (32) by using an electronic computer.

The ignition time, defined as the time to reach $\theta_1 = 10$, was found as a result of the solution.

In particular, for $\theta_H = 5, 10, 15, 20, 25, 30$ we have $\tau_3 = 15.0; 37.8; 62.2; 118; 175; 269$, respectively, which agrees with the results in [8].

Let us note that the problem of igniting a reagent by a heat flux has also been solved by the iteration – interpolation method. Mathematically this problem reduces to solving (28) under boundary conditions of the second kind [8]. Values which also agree with the results in [8] have been obtained for the ignition time.

4. Let us use the iteration–interpolation method to solve the equation

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial\theta}{\partial\tau} - 1 \tag{34}$$

under the boundary and initial conditions

$$\theta|_{x=0} = 0, \quad \theta|_{x=1} = 0, \quad \theta|_{\tau=0} = 0. \tag{35}$$

Let us partition the interval $0 < x < 1$ into $k + 1$ parts. Let us assign the zero approximation in the i-th interval in the form (13), where however, $x_i = ih$, $h = 1/(k+1)$, $\theta_0 = 0$ and $\theta_{k+1} = 0$. We obtain the first approximation in the i-th interval exactly as in (15) by considering $x_i = ih$ here.

From a condition analogous to the second condition in (16), we obtain a system of linear first-order inhomogeneous equations to determine $\theta_i(\tau)$

$$\dot{\theta}_{i-1} + 4\dot{\theta}_i + \dot{\theta}_{i+1} = \frac{6}{h^2}(\theta_{i-1} - 2\theta_i + \theta_{i+1}) + 6, \quad \theta_i|_{\tau=0} = 0. \tag{36}$$

The general solution of the system (36) is easily determined by the method of separation of variables [9] and is

$$\theta_i = \frac{x_i(1-x_i)}{2} + \sum_{s=1}^{k} C_s \sin \pi s x_i \exp(-\delta_s^2\tau),$$

$$\delta_s^2 = \frac{12 \sin^2 \dfrac{\pi s}{2(k+1)}}{h^2\left(2 + \cos \dfrac{\pi s}{k+1}\right)}. \tag{37}$$

Satisfying the initial condition (36) with (37), we obtain a system of linear inhomogeneous equations to determine C_s

$$\sum_{s=1}^{k} C_s \sin \pi s x_i = \frac{x_i(x_i - 1)}{2}, \quad i = 1, \ldots, k. \tag{38}$$

The results of the calculations are presented in Table 1.

The first number for each τ and x in Table 1 is the exact value θ_i and the second number is θ_i for $k = 9$. For $k = 3$ we have $\theta_2(0, 5) = 0.124285$ while the corresponding value obtained by using the method of lines [9] to the error $O(h^2)$ of the approximation is $\theta_2(0, 5) = 0.124881$. It follows from the results of the calculations that the accuracy of the approximate solution is somewhat higher than the accuracy of the solution obtained by the method of lines, and is raised as the number of "free" parameters increases.

Let us note that the iteration–interpolation method is also applicable for the solution of multidimensional boundary-value problems. In particular, if $\theta = \theta(x, y, \tau)$ and the domain of definition of the equation is a rectangle, then by integrating over the variable x, we obtain a system of linear partial differential equations to determine $\theta_i = \theta(y_i, \tau)$. Each of the linear equations can be solved exactly as (34), whereupon we again obtain a system of first-order ordinary differential equations to determine the values of the temperature $\theta_{ij} = \theta_{ij}(\tau)$ at the nodes, which can be solved by separation of variables or numerically.

5. Let us examine application of the proposed method to solve equations of hyperbolic type in the example of the problem of free vibrations of a string

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial \tau_1^2} \tag{39}$$

under the boundary and initial conditions

$$v(0, \tau) = 0, \quad v(0, a) = 0, \quad v(x, 0) = x(a - x), \quad \left.\frac{\partial v}{\partial \tau_1}\right|_{\tau_1=0} = 0. \tag{40}$$

Exactly as in the preceding section, we obtain the first approximation in the i-th interval in the form (15), where $\dot{A}_i$ and $\dot{B}_i$ should be replaced by $\ddot{A}_i$ and $\ddot{B}_i$, and $x_i = ih$.

From the merger conditions analogous to the second of conditions (16), we obtain a system of linear homogeneous ordinary differential equations to determine v_i

$$\ddot{v}_{i-1}+4\ddot{v}_i+\ddot{v}_{i+1}=\frac{6}{h^2}(v_{i-1}-2v_i+v_{i+1}),$$
$$\dot{v}_i(0)=0,\quad v_i(0)=x_i(a-x_i). \tag{41}$$

The general solution of this system of equations is easily determined by separation of variables [9] and is

$$v_i=x_i(a-x_i)+\sum_{s=1}^{k}\sin\frac{\pi s x_i}{a}(D_s\cos\delta_s\tau+E_s\sin\delta_s\tau). \tag{42}$$

Satisfying the initial conditions (41) with (42), we obtain a system of linear equations to determine the arbitrary constants D_s and E_s

$$\sum_{s=1}^{k}D_s\sin\frac{\pi s x_i}{a}=x_i(a-x_i),\qquad \sum_{s=1}^{k}\delta_s E_s\sin\frac{\pi x_i s}{a}=0, \tag{43}$$

from which it follows that $E_s = 0$ and $D_s \neq 0$.

For $a = 1$ and $k = 3$ we obtain $v_2(0, s) = 0.496648$ and for $k = 9$ we have numerical results presented in Table 2.

Keeping in mind that the disposition of the numbers in Table 2 is exactly the same as in the preceding table, we see that as before the accuracy of the method rises as k increases.

6. Since the solution of the corresponding boundary-value problem for an elliptic-type equation is obtained from the solution of the boundary-value problem for a parabolic-type equation as $\tau\to\infty$, then the iteration–interpolation method is also applicable in this case. As a simple illustration, let us consider the solution of the Poisson equation

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=-\alpha,\quad \alpha=\text{const} \tag{44}$$

under the boundary conditions

$$\theta|_{x=0,b}=0,\quad \theta|_{y=0,a}=0. \tag{45}$$

Partitioning the interval $0 < y < a$ into $k + 1$ parts just as before, then for $a = b = 1$, $\alpha = 1$ and $k = 1$ we find, $\theta_1(0.5) = 0.08211$, for $k = 3$ we obtain $\theta_2(0.5) = 0.07558$, and for $k = 9$ we obtain the results presented in Table 3.

Keeping in mind that the location of the numbers in Table 3 is the same as in the previous tables, we see that the accuracy of the method rises as the number of partitions increases.

7. It should be noted that the error of the approximation of interpolation schemes obtained by using the iteration–interpolation method is $O(h^3)$. Indeed, we have an error $O(h^2)$ for the approximation of $\partial\theta/\partial\tau$ or $\partial^2\theta/\partial x^2$ by linear functions. This error diminishes because of the iteration and becomes the quantity $O(h^3)$. If a second approximation had been found successfully, then the error in the approximation would have been $O(h^4)$. In general, the error in the approximation is $O(h^{2+n})$ for the n-th approximation. The results in Tables 1-3 do not contradict the assertion made.

Therefore, a large quantity of interpolation schemes can be constructed by using the iteration–interpolation method, and no rigid demands are imposed on the existence of higher-order derivatives for the desired function, as is done in solving the problems by the method of lines [9] or by difference methods, say.

The calculation of the second approximation is facilitated substantially by the fact that we obtain a system of ordinary differential equations, in which the derivatives of the desired functions enter linearly, to determine the desired functions at the interpolation nodes when solving both linear and nonlinear boundary-value problems.

Since the matrix of the coefficients for the derivatives is a Jacobi matrix [10], the derivatives can then easily be determined by using the method of factorization [11] or the Gauss method of elimination.

Having determined the derivatives, we eliminate them from the expressions for the profiles of the desired functions. Afterwards, the second approximation can be found exactly as has been done in Section 1.

Selection of the zero approximation is another source for decreasing the error in the approximation.

In the case of linear partial differential equations, the convergence of the method can apparently be proved theoretically since the examples show that the systems (36), (41) are analogous to corresponding systems obtained by using the method of lines, and the convergence of the method of lines has been proved [9].

NOTATION

$\tau = t/\varkappa$, t	time;
$\varkappa$	coefficient of temperature conduction;
$\theta = T - T_0$; T_0	absolute temperature of the medium at $t = 0$ and temperature on the boundaries of the system being studied;
$Q = W/\rho c$, W	quantity of energy introduced into the system at $t = 0$;
ρ	density;
c	specific heat;
$\mu = 0, 1, 2$ and $\alpha\mu = 2, 2\pi, 4\pi$	for plane, cylindrical and spherical symmetry, respectively;
x, y	space coordinates;
$V(x, \tau_1)$	displacement of a point with abscissa x at the equilibrium position;
$\tau_1 = at$; $a^2 = F/\rho_1$; F	thread tension;
ρ_1	linear thread density.

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